Some remarks about Cauchy integrals and fractal sets

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If μ is a finite Borel measure on the complex plane, then the Cauchy integral

$$C(\mu)(z) = \int_{\mathbf{C}} \frac{1}{z - \zeta} \, d\mu(\zeta) \tag{1}$$

defines a holomorphic function of z on the complement of the support of μ. For simplicity, let us restrict our attention to measures with compact support in $\mathbf{C}$, although one can also make sense of Cauchy integrals of measures with noncompact support and infinite mass under suitable conditions. In the classical situation where μ is supported on a nice curve, the Cauchy integral of μ has a jump discontinuity across the curve. For more regular measures, the Cauchy integral converges absolutely for every $z \in \mathbf{C}$ and defines a continuous function on the plane. Of course, this function is not holomorphic in any neighborhood of the support of μ. For any finite measure μ on $\mathbf{C}$, the Cauchy integral $C(\mu)(z)$ makes sense as a locally integrable function on $\mathbf{C}$, whose $\overline{\partial}$ derivative is a constant multiple of μ in the sense of distributions.

On nice regions in $\mathbf{C}$, the Cauchy integral formula can be used to recover arbitrary holomorphic functions from their boundary values, under suitable conditions. The Cauchy integral leads to a projection from general functions on the boundary to boundary values of holomorphic functions.

In more fractal situations, one might prefer to think of Cauchy integrals as a way of solving $\overline{\partial}$ problems, to make corrections to get holomorphic functions instead of producing them directly. One might view this as being more like several complex variables. A basic scenario would be to multiply a holomorphic function by a non-holomorphic function with some regularity, and to try to make some relatively small corrections to the product to get a holomorphic function. Perhaps the holomorphic function has nice boundary values and the other function is defined on the boundary, and the correction is intended to yield the boundary values of a holomorphic function. For example, the product of a holomorphic function and a rational function with poles in the interior may not be holomorphic, and this can be corrected with a finite-rank operator to get rid of the singularities.

Bergman spaces and projections could also be considered on the interior. It would still be regularity of the non-holomorphic functions up to the boundary that matters, though.

In $\mathbf{R}^n$, one can use Cauchy integrals associated to Clifford analysis. However, remember that the product of Clifford holomorphic functions is not necessarily holomorphic, because of noncommutativity of the Clifford algebra,

In any dimension, it is already somewhat interesting to look at Bergman spaces of locally constant functions on regions with infinitely many connected components.

There seem to be a lot of choices involved with fractals. This may be a bit disconcerting, but perhaps it is also just as well.

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